THE THEORY OF PLANE STRAIN OF PLASTICALLY ANISOTROPIC BODIES

(K TEORII PLOSKOI DEFORMATSII PLASTICHESKI ANIZOTROPNYKH TEL)

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The paper investigates the plane strain of an anisotropic rigid plastic body in the presence of stresses which correspond to the side of the yield prism suggested by Ivlev [1].

1. In the works of Ivlev [1], Hu [2] and Sawczuk [3] use was made of piecewise linear conditions for plastically anisotropic bodies. In [2,3], in the formulation of the yield conditions, the authors considered cases when the principal axes of stress and the principal axes of anisotropy were coincident at every point in an orthotropic body. There are, however, few problems for which these conditions are satisfied.

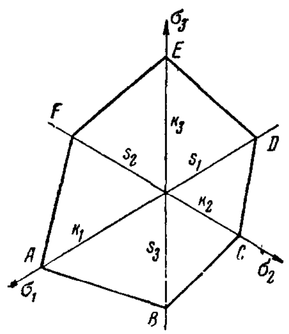

Fig. 1.

In the formulation of piecewise linear yield conditions for anisotropic bodies a change in the orientation of the principal axes of stress in an element of the body relative to a fixed system x, y, z is, in general, a factor which must be taken into account. The yield condition suggested by Ivlev [1] for bodies with plastic anisotropy of the most general kind do, in fact, take this factor into account. In the general case of plastic anisotropy the yield points in tension and compression are different and depend on the direction cosines given by the direction of the tension or compression relative to the axes x, y, z.

On the assumption that the yield condition is independent of the magnitude of the hydrostatic pressure and that the yield curve is not concave, the yield condition for an ideal plastic anisotropic body can be interpreted in the principal stress space as a six-sided prism, the sides of which are parallel to the line $\sigma_1 = \sigma_2 = \sigma_3$.

The yield condition (Fig. 1) is of the following form [1]:

$$\frac{\sigma_1-\sigma_2}{k_1}-\frac{\sigma_3-\sigma_2}{s_3}=1 \quad (AB), \qquad \frac{\sigma_3-\sigma_2}{k_3}-\frac{\sigma_1-\sigma_2}{s_1}=1 \quad (DE)$$

$$\frac{\sigma_2-\sigma_1}{k_2}-\frac{\sigma_3-\sigma_1}{s_3}=1 \quad (BC), \qquad \frac{\sigma_3-\sigma_1}{k_3}-\frac{\sigma_2-\sigma_1}{s_2}=1 \quad (EF) \qquad (1.1)$$

$$\frac{\sigma_2-\sigma_3}{k_2}-\frac{\sigma_1-\sigma_3}{s_1}=1 \quad (CD), \qquad \frac{\sigma_1-\sigma_3}{k_1}-\frac{\sigma_2-\sigma_3}{s_2}=1 \quad (FA)$$

Here

$k_1 = k_1(l_i)$, $k_2 = k_2(m_i)$, $k_3 = k_3(n_i)$ are the yield points in tension

$s_1 = s_1(l_i)$, $s_2 = s_2(m_i)$, $s_3 = s_3(n_i)$ are the yield points in compression in the direction of the principal axes of stress.

2. Let us consider the equilibrium of an anisotropic rigid plastic cylinder of infinite length, with generators parallel to the z-axis, under the action of a surface load which is constant along the generators. An isotropic body under such conditions would experience plane deformation. In a body which has the most general form of anisotropy plane deformation is not possible. Let us suppose that the body under consideration undergoes so-called pure plane deformation in such a way that the components of the stress tensor and the tensor of rate of strain are independent of z, and

$$\tau_{xz} = \tau_{yz} = 0, \qquad \varepsilon_z = \varepsilon_{xz} = \varepsilon_{yz} = 0 \qquad (2.1)$$

It follows from (2.1) that the direction of the z-axis is a principal direction for the stress tensor and the tensor of rate of strain.

Consequently, the orientation of the principal directions of stress relative to the x, y, z-system can be determined from the angle ϕ between the first principal direction and the x-axis. We note that the plastic state which satisfies condition (2.1) is given by points lying on the sides of the yield prism.

If $\sigma_1 > \sigma_3 > \sigma_2$ the side (FA) corresponds to the plastic state and has the equation

$$\frac{\sigma_1-\sigma_3}{k_1}-\frac{\sigma_2-\sigma_3}{s_2}=1 \qquad (k_1 = k_1(\varphi), \qquad s_2 = s_2(\varphi)) \qquad (2.2)$$

If we take the left-hand side of (2.2) as the plastic potential, we find that the condition $\epsilon_2 = 0$ can be satisfied only when $k_1(\phi) = s_2(\phi)$. It follows that certain conditions must be satisfied concerning the anisotropy of the body under consideration. If the body is orthotropic, then for the plane deformation given by (2.1) it is essential that the yield point $Y(\phi)$ in tension-compression satisfies the condition that $Y(\phi) = Y(\phi \pm 1/2\,\pi)$. A consideration of plane deformation of an orthotropic body with the yield condition used by Hill also leads to this fourfold symmetry of the yield point in tension-compression.

We note that the plane deformation corresponding to an edge of the yield prism which was considered by Ivlev in [1] does not cover all cases of the plane deformation of an anisotropic body, since in this case there are no limitations on the nature of the anisotropy, and transition in the limit to the isotropic case of plane deformation is not possible.

From now on we shall assume that the body we are considering satisfies the above requirements. In particular, the yield points in tension and compression for an orthotropic body are the same, and the condition of plasticity (2.2) in the case of plane deformation reduces to the form

$$\sigma_1 - \sigma_2 = Y(\varphi) \tag{2.3}$$

Transferring now to components in a system of Cartesian coordinates, we find that

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = Y^2(\varphi) \tag{2.4}$$

Equation (2.4) can easily be satisfied by substitution of expressions given by the transformation formulas

$$\sigma_x = p + \tfrac{1}{2} Y(\varphi)\cos 2\varphi, \qquad \sigma_y = p - \tfrac{1}{2} Y(\varphi)\sin 2\varphi, \qquad \tau_{xy} = \tfrac{1}{2} Y(\varphi)\sin 2\varphi \tag{2.5}$$

where

$$p = \frac{\sigma_x + \sigma_y}{2}, \qquad \varphi = \frac{1}{2}\tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{2.6}$$

Substituting Expressions (2.5) into the equilibrium equations, we obtain

$$\begin{aligned} &2\frac{\partial p}{\partial x} - (2Y\sin 2\varphi - Y'\cos 2\varphi)\frac{\partial \varphi}{\partial x} + (2Y\cos 2\varphi + Y'\sin 2\varphi)\frac{\partial \varphi}{\partial y} = 0 \\ &2\frac{\partial P}{\partial y} + (2Y\cos 2\varphi + Y'\sin 2\varphi)\frac{\partial \varphi}{\partial x} + (2Y\sin 2\varphi - Y'\cos 2\varphi)\frac{\partial \varphi}{\partial y} = 0 \end{aligned} \tag{2.7}$$

The set of equations (2.7) has two real families of orthogonal characteristics, the equations of which are

$$\frac{dy}{dx} = \frac{2Y\sin 2\varphi - Y'\cos 2\varphi \pm \sqrt{Y'^2 + 4Y^2}}{2Y\cos 2\varphi + Y'\sin 2\varphi} \tag{2.8}$$

Along the characteristics there exist the relations

$$p \pm G(\varphi) = \text{const} \qquad \left(G(\varphi) = \frac{1}{2}\int \sqrt{Y'^2 + 4Y^2}\, d\varphi\right) \tag{2.9}$$

Equations (2.5) through (2.9) coincide with the corresponding equations of Ivlev [1] (if the misprints in the latter are corrected), but they refer to the stresses on the face of the yield prism.

If we consider the left-hand side of the yield condition (2.4) as the plastic potential and make use of (2,6),we can establish the flow law in the following form:

$$\varepsilon_x = \lambda\left(\sigma_x - \sigma_y + \frac{Y'}{Y}\tau_{xy}\right), \qquad \varepsilon_y = \lambda\left(\sigma_y - \sigma_x - \frac{Y'}{Y}\tau_{xy}\right)$$

$$\gamma_{xy} = \lambda\left[4\tau_{xy} - \frac{Y'}{Y}(\sigma_x - \sigma_y)\right] \tag{2.10}$$

where ϵ_x, ϵ_y, γ_{xy} are the components of rate of strain.

It will be convenient here to introduce the function ψ as the angle between the first principal direction of rate of strain and the x-axis

$$\tan 2\psi = \frac{2Y\sin 2\varphi - Y'\cos 2\varphi}{2Y\cos 2\varphi + Y'\sin 2\varphi} \tag{2.11}$$

From (2.10), (2.11) and (2.5) we can easily find the set of equations which gives the velocity components v_x, v_y:

$$2\frac{\partial v_x}{\partial x} - \cot 2\psi\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) = 0, \qquad 2\frac{\partial v_y}{\partial y} + \cot 2\psi\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) = 0 \tag{2.12}$$

Equations (2.12) are of the hyperbolic type and their characteristics coincide with those of (2.7). After substitution of (2.11) the equation of the characteristics is considerably simplified and becomes

$$dy\,/\,dx = \tan\left(\psi \pm \frac{\pi}{4}\right) \tag{2.13}$$

Thus, the characteristics are slip-lines (by slip-lines we mean lines of maximum rate of shear strain). In view of the anisotropy they do not normally coincide with the lines of maximum shear stress.

If we now denote the components of the rates of strain along the slip lines α and β by u and v, we have from (2.12) the following equations:

$$\partial u / \partial s_\alpha = 0, \qquad \partial v / \partial s_\beta = 0 \tag{2.14}$$

which show that the rates of linear strain along the slip-lines are zero. After transforming (2.14) we obtain the well-known Geiringer relations

$$du - v d\psi = 0 \text{ along the line } \alpha \qquad dv + u d\psi = 0 \text{ along the line } \beta \tag{2.15}$$

3. It can easily be shown that there exist particular solutions to Equations (2.7) - the so-called integrals of the equations of the plane problem in the theory of plasticity. In general, the solution reduces to an examination of the boundary-value problems analogous to those for an isotropic body. The boundary conditions are given by the formulas

$$\sigma_n = p + \tfrac{1}{2} Y(\varphi) \cos 2(\varphi - \alpha), \qquad \tau_n = \tfrac{1}{2} Y(\varphi) \sin 2(\varphi - \alpha) \tag{3.1}$$

where σ_n and τ_n are the normal and tangential components of the stresses on the surface of the body, which is in a plastic state, and α is the angle between the normal to the surface and the x-axis.

We shall consider now the line of discontinuity of stress. From the condition of continuity of the stress components σ_n and τ_n on the lines of discontinuity we obtain the following relations:

$$p^+ + \tfrac{1}{2} Y(\varphi^+) \cos 2(\varphi^+ - \theta) = p^- + \tfrac{1}{2} Y(\varphi^-) \cos 2(\varphi^- - \theta) \tag{3.2}$$

$$\tfrac{1}{2} Y(\varphi^+) \sin 2(\varphi^+ - \theta) = \tfrac{1}{2} Y(\varphi^-) \sin 2(\varphi^- - \theta) \tag{3.3}$$

where θ is the angle between the normal to the line of discontinuity and the x-axis.

The angle θ can be found from (3.3) as follows:

$$\theta = \frac{1}{2} \tan^{-1} \frac{Y(\varphi^+) \sin 2\varphi^+ - Y(\varphi^-) \sin 2\varphi^-}{Y(\varphi^+) \cos 2\varphi^+ - Y(\varphi^-) \cos 2\varphi^-} \quad \text{or} \quad \theta = \tan^{-1} \frac{2[\tau_{xy}]}{[\sigma_x - \sigma_y]} \tag{3.4}$$

In contrast to the case of an isotropic body the line of discontinuity in general is nowhere the bisector of the angle formed by the slip-lines. In the case of an orthotropic body it can bisect this angle only when the line of discontinuity coincides with one of the principal axes of anisotropy of the material.

In our particular case of plane deformation the extremum theorems can be formulated in an analogous manner to the theorems for an isotropic body [5,6]. The assumptions that the yield curve is not concave and that the angular change at the yield point is limited enable us to establish two basic inequalities which formulate the extremum properties.

4. As an example we shall consider the problem of finding the limiting load for an obtuse-angled wedge under the action of a uniform pressure q applied to one side of the wedge. We shall assume that the material is orthotropic and behaves as a rigid plastic body. We shall take the principal axes of anisotropy of the material as the coordinate system x, y, z. The angle between the axis of symmetry of the wedge and the y-axis will be denoted by ϵ (Fig. 2).

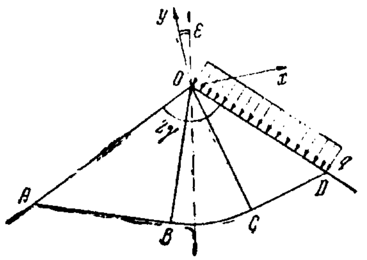

Fig. 2.

It is apparent that the triangular areas *ABO*, *COD* are in a state of uniform stress. These two areas are connected by the centered field *BOC*.

In ΔCOD $\phi_1 = \gamma - 1/2\,\pi - \epsilon$, $p_1 = -q + 1/2Y(\phi_1)$; the angle ψ_1 between *OC* and the x-axis can be found from Equation (2.11).

In ΔAOB $\phi_2 = -(\gamma + \epsilon)$, $p_2 = -1/2Y(\phi_2)$; the angle ψ_2 between *OB* and the x-axis can also be found from (2.11).

Thus, having found ψ_1 and ψ_2, we can construct the triangular areas *AOB* and *COD* and then join them with the centered fan *BOC*. The limiting load q^* can be found most easily from the condition that η is constant over the whole area.

It follows from (2.9) that

$$p_1 + G(\varphi_1) = p_2 + G(\varphi_2) \tag{4.1}$$

The limiting load is therefore

$$q^* = \frac{1}{2}\left[Y(\varphi_1) + Y(\varphi_2) + \int_{\varphi_2}^{\varphi_1} \sqrt{Y'^2 + 4Y^2}\,d\varphi\right] \tag{4.2}$$

In the case of an isotropic body we simply put $Y(\phi) = \sigma_s = \text{const}$, $\epsilon = 0$, $\phi = \psi$ in order to derive the well-known formula for the limiting load for an obtuse wedge [5].

In a similar way we can find very simply the limiting load required to compress a flat piece of material into a half-plane and solve several other problems.

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